

FREE VIBRATIONAL AND BUCKLING ANALYSIS OF C-SECTION LAMINATED COMPOSITE BEAM

PHÂN TÍCH DAO ĐỘNG TỰ DO VÀ ỔN ĐỊNH CỦA DẦM COMPOSITE LỚP VỚI MẶT CẮT CHỮ C

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ABSTRACT

Thin-walled composite beams with C-sections are widely used in aerospace, civil engineering, and marine structures due to the superior properties of laminated composite materials. Based on the first-order shear deformation plate theory and employing nine-noded rectangular isoparametric plate elements with five degrees of freedom per node, this study presents numerical results on the free vibration and buckling behavior of thin-walled composite beams with C-sections. The effects of fiber orientation and material properties on natural frequencies, buckling loads, and corresponding buckling mode shapes are investigated. This study provides a numerical database that supports structural design and analysis of lightweight composite beams, particularly in high-performance applications in the aerospace, automotive, and civil engineering industries.

Keywords: C-sections; Composite beam; Free vibration; Linear buckling; Mode shapes.

TÓM TẮT

Dầm composite thành mỏng với mặt cắt như hình chữ C được ứng dụng rộng rãi trong các lĩnh vực hàng không, xây dựng dân dụng và chế tạo tàu thuyền vì nhiều đặc tính ưu việt của vật liệu composite lớp. Dựa trên lý thuyết tấm biến dạng cắt bậc nhất và sử dụng phần tử tấm hình chữ nhật 9 nút với năm bậc tự do tại mỗi nút, nghiên cứu này trình bày một số kết quả số về đặc tính dao động tự do và ổn định uốn của dầm composite thành mỏng có tiết diện hình chữ C. Trong đó, ảnh hưởng của góc sợi composite và cơ tính vật liệu đến tần số dao động riêng, tải trọng tới hạn và các dạng mất ổn định tương ứng được xem xét. Nghiên cứu này đóng góp cơ sở dữ liệu số về tính toán thiết kế kết cấu cho các dầm composite nhẹ, đặc biệt trong các ứng dụng hiệu suất cao thuộc các ngành công nghiệp hàng không, ô tô và xây dựng dân dụng.

Từ khóa: Mặt cắt chữ C; Dầm composite; Dao động tự do; Ổn định tuyến tính; Mode dao động.

1. INTRODUCTION

Thin-walled composite beams with open C- or U-shaped cross-sections are widely employed in aerospace, civil engineering, and mechanical applications due to their high strength-to-weight ratio. However, the mechanical behavior of such beams is inherently complex, as it is influenced by the combined effects of bending, torsion, and local warping deformation of the cross-section. These effects significantly impact both the dynamic characteristics and the stability performance of the beams, particularly under varying boundary and loading conditions.

In recent decades, the analysis of free vibration and stability of thin-walled composite beams has attracted considerable attention in the research community. Although classical beam theories offer simplicity and computational efficiency, they generally fail to accurately capture the anisotropic behavior and interlaminar interactions typical of laminated composite materials, as well as the distortion and warping deformations characteristic of thin-walled open sections. To overcome these limitations, various advanced methods and theories have been developed, including the generalized beam theory, higher-order shear deformation models, and the finite element method (FEM), aiming to enhance the accuracy of dynamic and stability simulations.

Recent studies have emphasized the importance of accounting for shear deformation, torsional effects, and material inhomogeneity in the accurate prediction of natural frequencies and critical loads. For instance, Sheikh et al. [1] proposed a one-dimensional finite element model for multilayer thin-walled composite beams with both open and closed sections,

incorporating laminate configuration and out-of-plane deformations, and validated the model through numerical examples. Dey and Talukdar [2] utilized FEM to investigate the free vibration of channel-section beams considering warping deformation and compared their results with experimental data.

Hoang et al. in [3] study focused on the optimization of free vibration and lateral buckling stability of channel-section composite beams, using design variables such as flange width, web height, and fiber orientation. A beam element with seven degrees of freedom per node was employed in conjunction with a micro-genetic algorithm. Dennis et al. [4] developed an 8-d.o.f beam element based on an extended Vlasov theory following Andrade's formulation to overcome limitations in modeling tapered C-section beams. Tan et al. [5] examined the lateral buckling behavior of thin-walled functionally graded beams with symmetric I- and channel-sections using Vlasov's hypothesis, accounting for warping deformation and material gradation, combined with FEM.

Linh et al. [6] proposed an FEM model for thin-walled functionally graded beams with open sections, where the material properties vary along the cross-sectional boundary. The model allows for determining natural frequencies and mode shapes, while also investigating the influence of material distribution functions and geometry. Liu et al. [7] introduced a special dynamic stiffness model for rib-stiffened plate structures with arbitrary boundary conditions and section configurations, employing a modified Fourier series and the Wittrick–Williams algorithm for vibration analysis.



Xu et al. [8] focused on higher-order vibration modes of complex open-section thin-walled beams, which are highly sensitive to torsional and out-of-plane warping effects. Linh et al. [9] addressed the optimization of vibration and bending stability for FG beams with doubly symmetric I- and channel-sections, using a non-monotonic material distribution, a 14-d.o.f Vlasov-based beam element, and a differential evolution algorithm. Marzok and Waisman [10] employed the extended finite element method (XFEM) for buckling analysis of thin-walled beams, demonstrating that XFEM can yield accurate results with coarse meshes and reduced computational cost. Turkalj et al. [11] adopted the Timoshenko–Ehrenfest bending theory and a modified Vlasov torsion model with cubic Hermite interpolation to avoid shear locking in nonlinear stability analysis of multilayer composite beams.

In Vietnam, Binh and Trinh [12] developed a Matlab-based FEM program to study the effects of cutouts and boundary conditions on the natural frequencies and vibration modes of thin-walled box beams. Binh and Hien [13] used a 9-nodes, first-order shear deformation plate element (with 5-d.o.f per node) to analyze the impact of taper angle on the vibrational characteristics and transient deflection response of tapered composite thin-walled rectangular box beams.

In the present study, the free vibration and stability behavior of thin-walled composite beams with C- sections is investigated using a finite element model based on plate elements. Numerical results are presented to evaluate the effect of fiber orientation and laminated scheme on natural frequencies and buckling loads.

2. FORMULATION OF THE PROBLEM

2.1. The displacement field

According to the Reissner-Mindlin plate theory [14], the displacements (u , v , w) are referred to those of the mid-plane (u_0 , v_0 , w_0) as:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z\theta_x(x, y, t) \\ v(x, y, z, t) &= v_0(x, y, t) + z\theta_y(x, y, t) \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (1)$$

Here, t is time, θ_x and θ_y are the total rotations, ϕ_x and ϕ_y are the constant average shear deformations about the y and x -axes, respectively.

The z -axis is normal to the xy -plane that coincides with the mid-plane of the laminate positive downward and clockwise with x and y .

2.2. Finite element formulations

In laminated plate theories [14], the membrane $\{N\}$, bending moment $\{M\}$ and shear stress $\{Q\}$ resultants can be obtained by integration of stresses over the laminate thickness. The stress resultants-strain relations can be expressed in the form:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [A_{ij}] & [B_{ij}] & [0] \\ [B_{ij}] & [D_{ij}] & [0] \\ [0] & [0] & [F_{ij}] \end{bmatrix} \begin{Bmatrix} \{\epsilon^0\} \\ \{\kappa\} \\ \{\gamma^0\} \end{Bmatrix} \quad (2)$$

Where:

$$\begin{bmatrix} [A_{ij}] & [B_{ij}] & [D_{ij}] \end{bmatrix} = \sum_{k=1}^n \int_{h_{k-1}}^{h_k} \begin{bmatrix} [Q'_{ij}]_k \end{bmatrix} (1, z, z^2) dz \quad (i, j = 1, 2, 6) \quad (3)$$

$$[F_{ij}] = \sum_{k=1}^n f \int_{h_{k-1}}^{h_k} \begin{bmatrix} [C'_{ij}]_k \end{bmatrix} dz; \quad f = 5/6; \quad i, j = 4, 5 \quad (4)$$

n : Number of layers, h_{k-1} , h_k : The position of the top and bottom faces of the k^{th} layer.

$[Q'_{ij}]_k$ and $[C'_{ij}]_k$: Reduced stiffness matrices of the k^{th} layer (see [9]).

In the present work, nine-noded isoparametric quadrilateral element with five degrees of freedom per nodes is used.

For free and forced vibration analysis, the damping effect is neglected; the governing equations are [14]:

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \text{ or}$$

$$\{[M] - \omega^2[K]\}\{u\} = \{0\} \quad (4)$$

$$\text{And } [M]\{\ddot{u}\} + [K]\{u\} = f(t) \quad (5)$$

The eigenvalue type of buckling equation can be expressed as in the following steps by dropping force terms and conceded to:

$$\{[K] + \lambda_{CR}[K_G]\}\{\Delta d\} = \{0\} \quad (6)$$

Where, $[K_G]$ is the geometric stiffness matrix; $\{\Delta d\}$ is the displacement increment vector and λ_{CR} is the critical mechanical load at which the structure buckles; $[M]$, $[K]$ are the global mass matrix, stiffness matrix, respectively.

3. NUMERICAL RESULTS

3.1. The C-section laminated composite beam

Consider a C-section laminated composite beam is shown in Figure 1; the dimensions: $OO_1 = L = 2(\text{m})$, flange width: $OB = 0.2(\text{m})$, web depth: $OA = 0.1(\text{m})$, each ply

has a thickness of $0.25(\text{mm})$, laminated scheme $[\theta^\circ/-\theta^\circ]_{10S}$. The material parameters are set as follow: MAT1 (Graphite/Epoxy): $E_1(\text{GPa}) = 155$; $E_2, E_3(\text{GPa}) = 8.07$; $G_{12,13}(\text{GPa}) = 4.55$; $G_{23}(\text{GPa}) = 3.9$; $\nu_{12,13} = 0.22$; $\nu_{23} = 0.22$; $\rho(\text{kg/m}^3) = 1558$; MAT2 (Boron/Epoxy): $E_1(\text{GPa}) = 201$; $E_2, E_3(\text{GPa}) = 21.7$; $G_{12,13}(\text{GPa}) = 5.4$; $G_{23}(\text{GPa}) = 3.69$; $\nu_{12,13} = 0.17$; $\nu_{23} = 0.035$; $\rho(\text{kg/m}^3) = 1724$; MAT3(E-Glass/Epoxy): $E_1(\text{GPa}) = 41$; $E_2, E_3(\text{GPa}) = 10.04$; $G_{12,13}(\text{GPa}) = 4.3$; $G_{23}(\text{GPa}) = 3.45$; $\nu_{12,13} = 0.28$; $\nu_{23} = 0.28$; $\rho(\text{kg/m}^3) = 1860$.

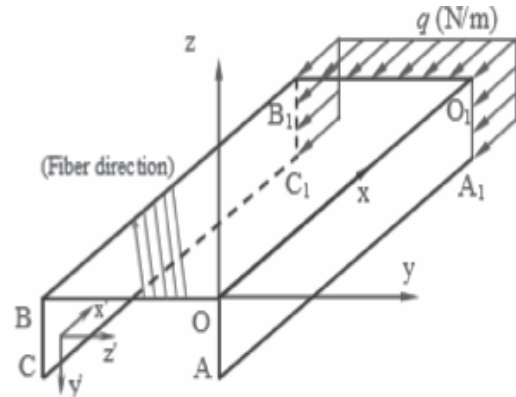


Figure 1. The C-section composite beam under uniform compression

Boundary conditions: At edges $x = 0$ are fixed and other edges are free.

3.2. Effect of fiber orientation on natural frequencies and buckling loads

In this subsection, to investigate the effect of fiber orientation on natural frequencies, the C-section beam made of MAT1 with laminate scheme $[\theta^\circ/-\theta^\circ]_{10S}$ are considered for different fiber orientation of $\theta = 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$ and $[0^\circ/90^\circ]_{10S}$. Figure 2 presents a comparison of the first five natural frequencies (ω_i) corresponding to different fiber orientations.



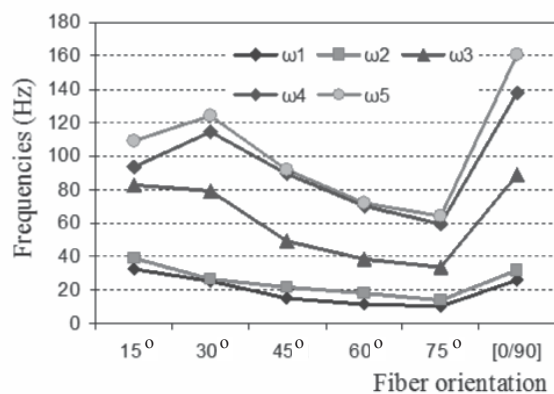


Figure 2. Effect of fiber orientation on natural frequencies

The first five buckling mode shapes of the beam with laminate scheme $[45^\circ/-45^\circ]_{10S}$ are shown in Figure 3.

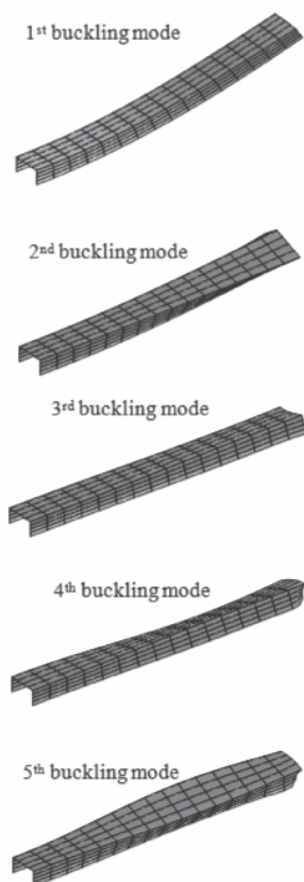


Figure 3. The first five buckling mode shapes (MATI, $[45^\circ/-45^\circ]_{10S}$)

The results shown in Figure 2 indicate that the natural frequencies of C-section composite beams are significantly influenced by the fiber orientation. As the fiber angle increases from 15° to 45° , all of the first five natural frequencies decrease, reaching their minimum values at approximately 45° . This trend reflects the reduction in axial stiffness due to the off-axis fiber effect. However, when the fiber angle increases further to 75° , the frequencies tend to rise again, attributed to the enhanced contribution from transverse stiffness. The $[0^\circ/90^\circ]_{10S}$ configuration yields the highest frequencies, demonstrating a well-balanced stiffness in both longitudinal and transverse directions. These findings highlight the critical importance of selecting an appropriate fiber orientation in the design of composite beams.

Figure 3 illustrates the obtained mode shapes, clearly showing a combination of bending, torsion, and warping deformations-characteristic of thin-walled structures with open cross-sections. The first mode is dominated by global bending in the principal plane. In the second and third modes, torsional effects become increasingly evident, particularly in the form of local twisting of the flange and web. The fourth and fifth modes reveal a complex interaction between bending and torsion, accompanied by pronounced warping at the section corners. These modes underscore the influence of fiber-induced anisotropy, especially in $[45^\circ/-45^\circ]_{10S}$. The early appearance of torsional and warping modes is a consequence of asymmetric stiffness and uneven stress distribution resulting from the fiber orientation, reducing the torsional resistance of the structure. This suggests that in the $[45^\circ/-45^\circ]_{10S}$ configuration, longitudinal stiffness is diminished while shear and torsional effects become more prominent under compressive loading.

Figure 4 presents a comparison of the first five buckling loads corresponding to different fiber orientations.

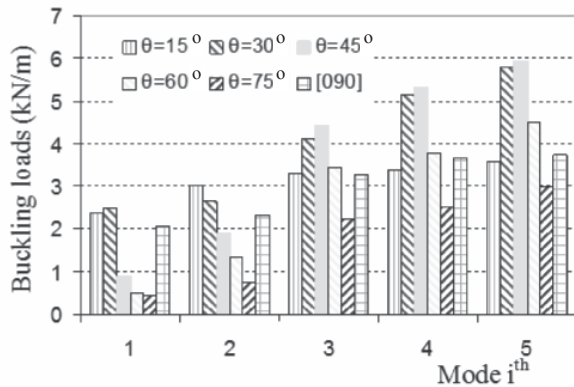


Figure 4. Effect of fiber orientation on buckling loads

Figure 4 illustrated that the critical buckling load reaches its maximum for the $[0^\circ/90^\circ]_{10S}$ configuration, where fibers are aligned parallel or perpendicular to the loading direction, thereby maximizing axial stiffness. In contrast, when the fiber orientation lies within $30^\circ - 60^\circ$, particularly at 45° , the buckling load significantly decreases due to the pronounced reduction in axial stiffness caused by the diagonal fiber effect.

3.3. Effect of material properties on natural frequencies and buckling loads

To investigate the effect of material properties on the buckling load, the natural frequencies, the C-section beam made of MAT1, MAT2 and MAT3, with laminated scheme of $[45^\circ/-45^\circ]_{10S}$ are considered. The first five natural frequencies for various material properties are compared and illustrated in Figure 5.

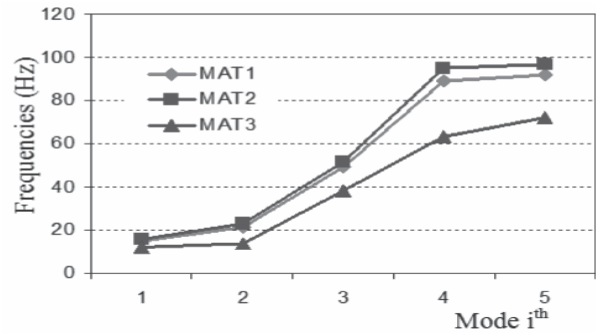


Figure 5. Effect of material properties on natural frequencies

Figure 5 shows that among the three materials studied, Boron/Epoxy (MAT2) exhibits the highest natural frequencies across all modes, followed by Graphite/Epoxy (MAT1) and E-Glass/Epoxy (MAT3). This outcome reflects differences in longitudinal modulus and stiffness-to-mass ratio. With a high E_1 modulus and relatively low density, Boron/Epoxy provides superior overall stiffness without a significant increase in weight, leading to enhanced dynamic performance. Conversely, E-Glass/Epoxy has a higher density but lower stiffness, resulting in reduced natural frequencies. Therefore, material selection should be carefully considered according to the specific dynamic performance requirements of the structure.

Figure 6 presents a comparison of the first five buckling loads corresponding to different material properties.

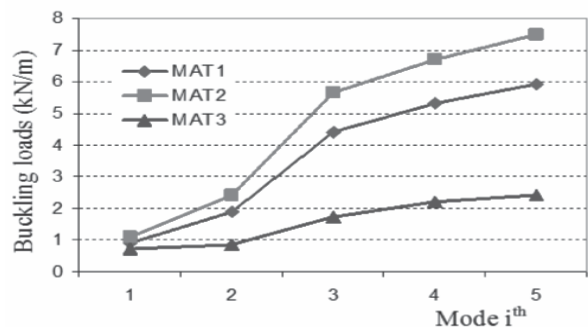


Figure 6. Effect of material properties on buckling loads

Figure 6 illustrated that the critical buckling load is strongly affected by material properties. Boron/Epoxy again demonstrates superior performance, achieving the highest buckling loads across all modes due to its high longitudinal stiffness, which delays the onset of instability under compressive loads. In contrast, the E-Glass/Epoxy beam exhibits the lowest buckling loads, indicating its limitations in high-compression applications. These findings reaffirm the significant role of material characteristics, not only in terms of global stiffness but also in anisotropy and lamination structure, in enhancing the buckling resistance of composite beams.

4. CONCLUSIONS

This study presented a finite element analysis of the free vibration and linear buckling behavior of thin-walled composite C-section beams using a nine-node shell element based on the Reissner–Mindlin theory. The numerical results clearly demonstrate that fiber orientation has a substantial impact on both natural frequencies and critical buckling loads. For the specific beam configuration considered, the $[0^\circ/90^\circ]_{10S}$ fiber arrangement optimizes both vibrational and buckling performance. Boron/Epoxy, with its high modulus and favorable stiffness-to-weight ratio, offers superior performance in both domains. The nine-node plate element proved effective in capturing the combined effects of bending, torsion, and shear in thin-walled open-section beams. The numerical findings provide valuable insights for engineers involved in the design and optimization of high-performance composite beams. ❖

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